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ON NONLINEAR ARTIFICIAL VISCOSITY, DISCRETE MAXIMUM PRINCIPLE AND HYPERBOLIC CONSERVATION LAWS*

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Abstract.

A finite element method for Burgers' equation is studied. The method is analyzed using techniques from stabilized finite element methods and convergence to entropy solutions is proven under certain hypotheses on the artificial viscosity. In particular we assume that a discrete maximum principle holds. We then construct a nonlinear artificial viscosity that satisfies the assumptions required for convergence and that can be tuned to minimize artificial viscosity away from local extrema.

The theoretical results are exemplified on a numerical example.

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1 Introduction.

In this note we will revisit some of our results on discrete maximum principles for finite element methods [2, 3, 4] and show how they may be applied for the analysis of one dimensional non-linear conservation laws. It is well known that any centered finite difference scheme or the standard Galerkin method will exhibit violent spurious oscillations close to shocks due to the conservation properties (no entropy is produced). Different techniques have been proposed to solve this problem. In the Galerkin framework, stabilized finite element methods using shock-capturing have been investigated by Johnson and Szepessy in [10] and by Szepessy in [15]. In these works convergence to entropy solutions for scalar conservation laws has been proved both in the one and two dimensional case. Essentially the streamline-diffusion/shock capturing (SD/SC) method works because one term (SD) gives L^2-control of the low order residual and the other term (SC) gives a stronger local control close to shocks (hence the name). Success is obtained by a combination of a global anisotropic diffusion (in the streamline

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direction) and a residual based isotropic diffusion the coefficient of which is $O(h)$ in the vicinity of shocks.

In the finite volume or finite difference community on the other hand oscillation free solutions have been obtained by the introduction of so call slope limiters or flux limiters that will assure that oscillations remain bounded. For a discussion of slope limiter high resolution schemes of finite difference or finite volume type, see the monographs by Leveque [12] or by Godlewski and Raviart [8] and references therein. The discontinuous Galerkin method is often considered as a natural generalization of finite volume methods in a Galerkin framework, and it has been considered in the case of scalar conservation laws in the work by Jaffre, Johnson and Szepessy [9] and the works of Cockburn and Shu, see for instance [7].

The aim of the present paper is to revisit the concept of nonlinear artificial viscosity for conservation laws in finite element methods using the ideas from the theory of discrete maximum principles, in the simple one dimensional case. In particular we give sufficient conditions on the viscosity parameter for convergence to entropy solutions. More precisely we show that under the following hypotheses on the nonlinear artificial viscosity, $\epsilon(u_h)$, (where u_h denotes the finite element solution associated to the mesh-size h)

(H1) $\epsilon(u_h)\partial_x u_h$ is locally Lipschitz continuous,
(H2) the bounds $0 \leq \epsilon(u_h) \leq Ch\|u_h\|_{L^\infty(\Omega)}$ hold,
(H3) $\epsilon(u_h)$ is sufficiently strong to make the method enjoy a discrete maximum principle,

the approximating sequence of finite element solutions converges to the unique entropy solution of the conservation law.

We propose an analysis for the standard Galerkin method using piecewise affine continuous approximation and shock capturing artificial viscosity. Compared to the streamline-diffusion/shock capturing method proposed in [10] our method uses only a nonlinear artificial viscosity or shock capturing term and no least squares stabilization of the residual. Moreover, we give an example of a function of $\epsilon(u_h)$ that satisfies (H1)–(H3), and has a weak consistency property, in fact, the viscosity can be tuned to be as small as desired away from local extrema. In our case the shock-capturing term is based not on the low order residual, but on the jump of the gradient over element edges.

In the next section we will introduce the concept of the DMP-property. This property of the bilinear (or semilinear) form of the weak formulation yields sufficient conditions for discrete maximum principles in finite element methods and is the cornerstone of our analysis. As we shall see the DMP-property of the semilinear form associated to a problem provides a sufficient condition for the numerical solution to be local extremum diminishing (LED). This strong monotonicity property is then used to prove that the sequence of finite element solutions to the one dimensional Burgers' equation converges to the weak entropy solution of the conservation law.

An outline of the paper is as follows, first, in Section 2 we present the abstract theory for discrete maximum principles, we then consider the case of non-linear conservation laws with strictly convex flux function in Section 3 and prove convergence to entropy solutions under the above hypotheses. In Section 4 we then construct a class of artificial viscosities where a parameter allows to control the spread of the artificial viscosity. The convergence theory applies uniformly in the parameter and we compare this artificial viscosity with the classical von Neumann–Richtmyer artificial viscosity. Finally in Section 5 we show the performance of the method on two simple model cases. One includes a shock wave and a rarefaction wave while the other has a smooth solution.

2 The DMP-property.

Let Ω be some open, simply connected domain and I some time interval defining a space time domain $Q = \Omega \times I$. Consider the following abstract problem on Ω: find $u \in L(0,T;V)$ such that

$$(\partial_t u, v)_Q + \int_0^T a(u;v)\ \mathrm{d}t = (g,v)_Q, \quad \forall v \in L(0,T;W) \tag{2.1}$$

with $u(0) = u_0$ and its finite element discretization: find $u_h \in V_h$ such that

$$(\partial_t u_h, v_h)_h + a_h(u_h;v_h) = (g, v_h), \quad \forall v_h \in V_h \text{ and } \forall t \in I \tag{2.2}$$

with $(u_h(0), v_h)_\Omega = (u_0, v_h)_\Omega$ for all $v_h \in V_h$. Here, V and W are two function spaces and V_h is the finite element space consisting of piecewise affine continuous functions defined on some nonoverlapping, conforming triangulation $\mathcal{T}_h$ of Ω, $a(\cdot;\cdot)$ is a semi-linear form with some discrete counterpart $a_h(\cdot;\cdot)$, $(\cdot,\cdot)_X$ denotes the $L^2(X)$-scalar product and $(\cdot,\cdot)_h$ denotes the $L^2(\Omega)$-scalar product evaluated using nodal quadrature. The corresponding norms are denoted by $\|\cdot\|_X = (\cdot,\cdot)_X^{\frac{1}{2}}$ and $\|\cdot\|_h = (\cdot,\cdot)_h^{\frac{1}{2}}$. We will use the notation $[\![w]\!]_e$ for the jump of w over element boundary e in one space dimension, defined by

$$\lim_{\varepsilon\to 0} w(x_e+\varepsilon) - w(x_e - \varepsilon)$$

and $\{w\}_e$ for the average of w over element boundary e, defined by

$$\lim_{\varepsilon\to 0} \frac{1}{2}\left(w(x_e+\varepsilon) + w(x_e-\varepsilon)\right).$$

By v_i we will denote the basis functions in V_h such that $v_i(x_j) = \delta_{ij}$ for all nodes x_j in the mesh, where δ_{ij} denotes the Kronecker delta funtion. The support of v_i will be denoted Ω_i. We assume that the continuous Cauchy-problem (2.1) problem is well-posed. Local well-posedness for the discrete problem (2.2) is obtained provided $a_h(u_h;v)$ is locally Lipschitz continuous. Very often the analysis of (2.1) relies on a maximum principle of some kind. Unfortunately such maximum principles are inherited by the discrete version of the problem only in particular cases [6, 18, 16, 2], such methods are said to satisfy a discrete maximum principle (DMP). In particular in the important case of hyperbolic conservation laws the finite element solution obtained using a linear scheme does not satisfy a DMP unless first order artificial viscosity is added.

Because of the nonlinearity of a_h, a DMP for (2.2) cannot be proved by showing that the stiffness matrix is an M-matrix. A framework for the study of discrete maximum principles for semilinear forms was proposed in [4].

DEFINITION 2.1. *We say that the semi-linear form $a_h(u_h; v)$ has the* strong DMP-property *if the following holds true: $\forall u_h \in V_h$ and for all interior vertex x_i, if u_h is locally minimal (resp. maximal) on vertex x_i over macro-element Ω_i ($u_h(x_i) \leq u_h(x)$, $\forall x \in \Omega_i$) then there exists $\alpha_K > 0$ such that*

$$a_h(u_h; v_i) \leq - \sum_{K \in \Omega_i} \alpha_K |\nabla u_h|_K|. \tag{2.3}$$

(resp. $a_h(u_h; w_i) \geq \sum_{K \in \Omega_i} \alpha_K |\nabla u_h|_K|$)

DEFINITION 2.2. *We say that the semi-linear form $a(u_h; v)$ has the* weak DMP-property *if it satisfies the criterion of the strong DMP-property for local minima under the additional assumption that the local minimum is negative.*

Following the ideas of [2, 3, 4], the DMP-property may be used to prove discrete maximum principles for time dependent or stationary convection–diffusion-reaction equations discretized with piecewise affine finite elements. For completeness we sketch the argument for a particular case. Assume that we solve the problem find $u_h \in V_h^0$ (where V_h^0 is the space of piecewise affine continuous functions satisfying homogeneous Dirichlet boundary conditions) such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h^0, \text{ with } f \geq 0 \text{ in } \Omega.$$

Suppose that for the continuous problem a maximum principle holds such that $u \geq 0$. We then wish to prove that u_h also satisfies $u_h \geq 0$, i.e. that the discrete method inherits the maximum principle of the continuous problem. We assume that $a_h(\cdot; \cdot)$ enjoys the weak DMP-property. Suppose now that there exists a node S_i such that $u_h(S_i) < 0$ and $u_h(S_i)$ is a local minimum. Testing with v_i we have

$$a_h(u_h, v_i) \leq - \sum_{K \in \Omega_i} \alpha_K |\nabla u_h|_K|.$$

However since $a_h(u_h, v_i) = (f, v_i) \geq 0$, the only way this can hold true is if $\nabla u_h = 0$ for all $K \in \Omega_i$ and hence S_i is not a strict local minimum. Since we can repeat the argument for all nodes up to the boundary, the minimum is taken on the boundary and hence $u_h \geq 0$ in Ω.

In general one may say that the weak DMP-property is associated to positivity whereas the strong DMP-property may be associated to monotonicity, as we shall see in Section 5.

3 The Burgers' equation: convergence to entropy solutions.

To fix the ideas, from now on, we will consider the case of the homogeneous Burgers' equation set on an interval Ω of the real line with periodic boundary conditions and in the time interval $I = (0, T)$. The initial data have bounded variation, $u_0 \in \mathrm{BV}$. The source term is zero, $g = 0$, and the bilinear form is given by

$$a(u; v) = -\left(\frac{u^2}{2}, \partial_x v\right)_\Omega.$$

The discrete form is simply $a(\cdot\,;\cdot)$ with an artificial viscosity term added.

$$a_h(u_h; v_h) = -\left(\frac{u_h^2}{2}, \partial_x v_h\right)_\Omega + (\epsilon(u_h)\partial_x u_h, \partial_x v_h)_\Omega. \tag{3.1}$$

We assume that the $\epsilon(u_h)$ satisfies the hypotheses (H1)–(H3) of the introduction. Recall that the artificial viscosity coefficient $\epsilon(u_h)$ is assumed to be Lipschitz continuous and satisfy the bounds $0 \le \epsilon(u_h) \le Ch\|u_h\|_{L^\infty(\Omega)}$. Moreover to satisfy assumption (H3) we assume that the form $a_h(\cdot\,;\cdot)$ enjoys the strong DMP-property of Definition 2.1. We let V_h be the space of piecewise affine continuous functions defined on a uniform mesh with N elements, with the nodes $\{x_i\}_{i=0}^N$ and local mesh size $h = |\Omega|N^{-1}$. It follows that the discrete scalar product writes $(u, v)_h = \sum_{i=0}^N u(x_i)v(x_i)h$.

Since the form $a_h(\cdot\,;\cdot)$ (3.1) is locally Lipschitz continuous the discrete problem admits a local solution and using the lemmata below this solution may be extended to all time. To prove that the sequence of solutions $\{u_h\}_h$ converges to the unique entropy solution we typically need an L^∞-stability bound to assure weak-* convergence and a BV estimate uniform in h and in time. First of all we note that we have the following energy stability

LEMMA 3.1. *Let u_h be the solution of* (2.2) *with the semilinear form* (3.1). *Then there holds for $h > 0$*

$$\frac{1}{2}\|u_h(T)\|_h^2 + \left\|\epsilon(u_h)^{\frac{1}{2}}\partial_x u_h\right\|_Q^2 = \frac{1}{2}\|u_h(0)\|_h^2.$$

PROOF. Immediate by taking $v_h = u_h$ in (2.2) and integrating in time. Note that

$$(u_h\partial_x u_h, u_h)_\Omega = 0,\ \forall t \in \mathbb{R}_+$$

thanks to the periodic boundary conditions. □

REMARK 3.1. In fact the theory developed in the following section may be extended to any flux function $f(u)$ that is strictly convex, $C^2(\mathbb{R})$ and such that there exists $c_f \in \mathbb{R}$ such that $f(u) = c_f f'(u)u$, for all u. More precisely $f(u) = u^{2l}$ where $l \ge 1$ is an integer.

LEMMA 3.2. *Let u_h be the solution of* (2.2) *with* $g = 0$ *and with the semilinear form* (3.1) *satisfying the strong DMP-property. Then any local extremum of* $|u_h|$ *is decreasing in time (LED). As an immediate consequence* $\|u_h\|_{L^\infty(Q)} \leq \|u_h(0)\|_{L^\infty(\Omega)}$.

PROOF. Assume that there is a local maximum in the node x_i. Take $v_h = v_i$ in (2.2). Since the lumped mass is used for the time derivative term we have

$$\partial_t u_h(x_i, t) = -h^{-1} a_h(u_h, v_i) \leq 0$$

where the last inequality is a consequence of the strong DMP-property assumption. Hence any local maximum must be decreasing. The result for local minima follows in the same fashion. The L^∞-bound follows by the LED property. □

The only way the total variation can increase for a solution that is LED is by the appearance of new local extrema. Hence the importance of the following lemma.

LEMMA 3.3. *(Conservation of local extrema) Let u_h be the solution of* (2.2) *with* $g = 0$ *and with the semilinear form* (3.1) *satisfying the strong DMP-property. The number of local extrema in* $u_h(\cdot, t)$ *is smaller than or equal to the number of local extrema in* $u_h(0)$.

PROOF. Local extrema can appear only in a process where the gradient in one cell changes sign with respect to the gradient in the neighbouring cells. Clearly for this to happen the value of the gradient has to vanish at some time t, since u_h is C^0 in space and time for each fixed h. Consider therefore an element K with $\partial_x u_h = 0$ and such that its neighbouring elements K' and K'' have gradients with the same sign (otherwise $u_h|_K$ is already a local extremum). Let x_i denote the left endpoint of K and x_{i+1} the right endpoint, with corresponding testfunctions v_i and v_{i+1}. We assume that $\partial_x u_h|_{K'} \geq 0$ and $\partial_x u_h|_{K''} \geq 0$ (see Figure 3.1). If $\partial_t \partial_x u_h|_K \geq 0$ then no new local extremum can appear.

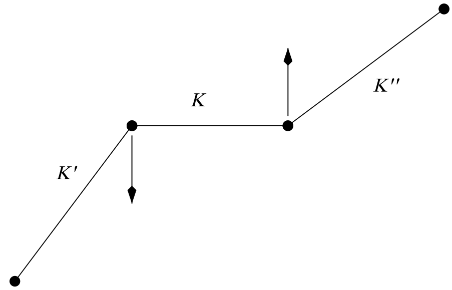

Figure 3.1: Illustration of the situation described in Lemma 3.3.

Consider therefore the gradient on element K.

$$
\begin{aligned}
\partial_t \partial_x u_h|_K &= \partial_t h^{-1}(u_h(x_{i+1},t) - u_h(x_i,t)) \\
&= h^{-1}(\partial_t u_h(x_{i+1},t) - \partial_t u_h(x_i,t)) \\
&= h^{-2}(a_h(u_h;v_{i+1}) - a_h(u_h;v_i)).
\end{aligned}
$$

However since $a_h(\cdot;\cdot)$ has the DMP-property $a_h(u_h;v_{i+1}) \geq 0$ and $a_h(u_h;v_i) \leq 0$. It follows that $\partial_t\partial_x u_h|_K \geq 0$ and in the same fashion one may prove that $\partial_t\partial_x u_h|_K \leq 0$ when $\partial_x u_h|_{K'} \leq 0$ and $\partial_x u_h|_{K''} \leq 0$. Hence no new local extrema can be created. □

REMARK 3.2. Note that this result does not imply that a node may not become a local extremum during the time evolution. It says that no new local extremum can be created, but sets no limits to the transport of existing maximum or minimum points. However the same technique may be used to show that as a local extremum moves from one node to the next it can never grow. Details are left to the reader.

LEMMA 3.4. *Let u_h be the solution of (2.2) with $g = 0$ and with the semilinear form (3.1) satisfying the strong DMP-property. Then there holds*

$$\partial_t TV(u_h) \leq 0.$$

PROOF. For $u_h \in V_h$, $TV(u_h) = \sum_{j=0}^{N} |u_h(x_j) - u_h(x_{j-1})|$, with the convention that $x_{-1} = x_N$. Let $K^+_{x_j}$ ($K^-_{x_j}$) denote the element whose left (right) endpoint is x_j. At each time t we may extract the ordered sets of positions of those local maxima of $u_h\{\hat{x}_j\}_{j=0}^M$ for which $\partial_x u_h|_{K^+_{\hat{x}_j}} < 0$, and local minima $\{\breve{x}_j\}_{j=0}^M$ for which $\partial_x u_h|_{K^-_{\breve{x}_j}} < 0$. M denotes the cardinality of these sets. By Lemma 3.3 M is bounded from above uniformly in h and t. The criteria on the gradient are chosen so as to include only one end point of constant portions of the solution that are local extrema.

Assume now that the sets are numbered in such a way that $\hat{x}_{j-1} < \breve{x}_j < \hat{x}_j$ for all j. By construction this is always possible.

The total variation may then be written

$$TV(u_h(t)) = \sum_{j=0}^{M} (u_h(\hat{x}_j,t) - u_h(\breve{x}_{j+1},t) + u_h(\hat{x}_j,t) - u_h(\breve{x}_j,t))$$

with the convention that $\breve{x}_{M+1} = \breve{x}_0$ (by periodicity). We conclude by deriving in time and noting that $\partial_t u_h(\hat{x}_j,t) \leq 0$ and $\partial_t u_h(\breve{x}_j,t) \geq 0$ by the LED property of Lemma 3.2. □

LEMMA 3.5. *Let u_h be the solution of (2.2) with $g = 0$ and with the semilinear form (3.1) satisfying the strong DMP-property. Then for all $h > 0$ there holds*

$$\|\partial_t u_h(t)\|_{L^1(\Omega)} + \|\partial_x u_h(t)\|_{L^1(\Omega)} \leq C, \quad \forall t \in \mathbb{R}_+ \tag{3.2}$$

and

$$\|u_h(\cdot,t_1)-u_h(\cdot,t_2)\|_{L^1(\Omega)} \le C|t_1-t_2|, \quad \forall t_1,t_2 \in \mathbb{R}_+ \tag{3.3}$$

with C independent of h.

Proof. Since $\partial_t TV(u_h(t)) \le 0$ and $\|\partial_x u_h(\cdot,t)\|_{L^1(\Omega)} = TV(u_h(t))$ we immediately conclude that $\|\partial_x u_h(\cdot,t)\|_{L^1(\Omega)} \le C$. Consider now

$$\begin{aligned}\|\partial_t u_h(\cdot,t)\|_{L^1(\Omega)} &\le \sum_i |\partial_t u_h(x_i,t)|h \\ &= \sum_i |a_h(u_h;v_i)| \le \|u_h\|_{L^\infty(\Omega)} \sum_i \|\partial_x u_h(\cdot,t)\|_{L^1(\Omega_i)} \\ &\le c\|u_h\|_{L^\infty(\Omega)}\|\partial_x u_h(\cdot,t)\|_{L^1(\Omega)}.\end{aligned}$$

For inequality (3.3) we note that

$$\begin{aligned}&\|u_h(\cdot,t_1)-u_h(\cdot,t_2)\|_{L^1(\Omega)} \le \sum_i |u_h(x_i,t_1)-u_h(x_i,t_2)|h \\ &\quad = \sum_i \Big|\int_{t_2}^{t_1} \partial_t u_h(x_i,t)\mathrm{d}t\Big|h = \sum_i \Big|\int_{t_1}^{t_2} a_h(u_h(\cdot,t),v_i)\ \mathrm{d}t\Big| \\ &\quad \le \int_{t_1}^{t_2} \sum_i |a_h(u_h(\cdot,t),v_i)|\ \mathrm{d}t \le \|u_h\|_{L^\infty(Q)} \int_{t_1}^{t_2} \sum_i \|\partial_x u_h(\cdot,t)\|_{L^1(\Omega_i)}\ \mathrm{d}t \\ &\quad \le C|t_1-t_2|\|u_h\|_{L^\infty(Q)} \max_{t\in[t_1,t_2]} \|\partial_x u_h(\cdot,t)\|_{L^1(\Omega)}\end{aligned}$$

and we conclude applying the first inequality. □

Combining the results of Lemma 3.2, Lemma 3.4 and Lemma 3.5 we may conclude that

$$\|u_h(t)\|_{L^\infty(\Omega)} + TV(u_h(t)) \le C, \quad \forall t \in \mathbb{R}_+$$

and

$$\|u_h(t_1)-u_h(t_2)\|_{L^1(\Omega)} \le C|t_1-t_2|, \quad \forall t_1,t_2 \in \mathbb{R}_+$$

with C independent of h. It then follows by Helly's theorem [11, Theorem A.3, p. 261] that we may extract a subsequence from $\{u_h\}_h$ such that $u_h \to u$ for almost all (x,t) and $u_h \to u$ in $L^1(\Omega)$ for all $t \in \mathbb{R}_+$. Moreover the limit function u satisfies

$$\|u(t)\|_{L^\infty(\Omega)} + TV(u(t)) \le C, \quad \forall t \in \mathbb{R}_+$$

and

$$\|u(t_1)-u(t_2)\|_{L^1(\Omega)} \le C|t_1-t_2|, \quad \forall t_1,t_2 \in \mathbb{R}_+.$$

We now recall a discrete commutator property introduced by Johnson and Szepessy [10] (see also Bertoluzza [1]). Let $\mathcal{I}_h$ denote the standard nodal interpolant.

LEMMA 3.6. *Let $u_h \in V_h$ and $\phi \in C^\infty(\Omega)$ then there holds*

$$\|u_h\phi - \mathcal{I}_h u_h\phi\|_{L^1(\Omega)} \leq Ch\|u_h\|_{L^1(\Omega)}\|\phi\|_{W^{1,\infty}(\Omega)} \tag{3.4}$$

$$\|\partial_x(u_h\phi - \mathcal{I}_h u_h\phi)\|_{L^2(\Omega)} \leq C\|u_h\|_{L^2(\Omega)}\|\phi\|_{W^{1,\infty}(\Omega)} \tag{3.5}$$

and

$$\|u_h\phi - \mathcal{I}_h u_h\phi\|_{L^\infty(\Omega)} \leq Ch\|u_h\|_{L^\infty(\Omega)}\|\phi\|_{W^{1,\infty}(\Omega)} \tag{3.6}$$

PROOF. See [10, 1]. □

THEOREM 3.7. *(Convergence to weak solutions) The solutions $\{u_h\}_h$ of (2.2) with a semilinear form (3.1) satisfying the strong DMP-property and $g = 0$ converges to a weak solution of (2.1).*

PROOF. We must show that in the limit u_h satisfies

$$(u_0, \phi(0))_\Omega - (u_h, \partial_t\phi)_Q - \left(\frac{u_h^2}{2}, \partial_x\phi\right)_Q = 0$$

for all $\phi \in C^\infty(Q)$ with $\phi(0,t) = \phi(1,t)$ and $\phi(x,T) = 0$. Using the formulation (2.2) we have after integration by parts

$$\begin{aligned}
&\left|(u_h(0), \phi(0))_\Omega - (u_h, \partial_t\phi)_Q - \left(\frac{u_h^2}{2}, \partial_x\phi\right)_Q\right| \\
&\quad \leq \left|\int_\Omega (u_h(0)\phi(0) - \mathcal{I}_h(u_h(0)\mathcal{I}_h\phi(0)))\,\mathrm{d}x\right| \\
&\quad\quad + \left|\int_0^T \int_\Omega (\partial_t u_h\phi - \mathcal{I}_h(\partial_t u_h \mathcal{I}_h\phi))\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\quad + \left|\int_0^T \int_\Omega \partial_x \frac{u_h^2}{2}(\phi - \mathcal{I}_h\phi)\,\mathrm{d}x\,\mathrm{d}t\right| + \left|(\epsilon(u_h)\partial_x u_h, \partial_x \mathcal{I}_h\phi)_Q\right| \\
&\quad = I + II + III + IV.
\end{aligned}$$

We proceed to bound I–IV term by term. First note that by the definition of the nodal interpolant and by Lemma 3.6, (3.4) we have for term I and II respectively

$$\begin{aligned}
I = \left|\int_\Omega (u_h(0)\phi(0) - \mathcal{I}_h(u_h(0)\phi(0)))\,\mathrm{d}x\right| &\leq Ch\|u_h(0)\|_{L^1(\Omega)}\|\phi\|_{W^{1,\infty}(\Omega)} \\
&\leq Ch\|u_0\|_{L^2(\Omega)}\|\phi\|_{W^{1,\infty}(\Omega)}
\end{aligned}$$

and

$$II = \Big| \int_0^T \int_\Omega (\partial_t u_h \phi - \mathcal{I}_h(\partial_t u_h \phi)) \; \mathrm{d}x \, \mathrm{d}t \Big| \le Ch \|\partial_t u_h\|_{L^1(Q)} \|\phi\|_{L^\infty(I;W^{1,\infty}(\Omega))}.$$

For term II we use interpolation to obtain

$$III \le \|u_h\|_{L^\infty(Q)} \|\partial_x u_h\|_{L^1(Q)} Ch \|\phi\|_{L^\infty(I;W^{1,\infty}(\Omega))}.$$

Finally we bound III by using the stability estimate of Lemma 3.1, the assumption (H2) on $\epsilon(u_h)$ and the H^1-stability of the nodal interpolant

$$\begin{aligned} IV &\le (\epsilon(u_h)\partial_x u_h, \partial_x u_h)_Q^{\frac{1}{2}} (\epsilon(u_h)\partial_x \mathcal{I}_h \phi, \partial_x \mathcal{I}_h \phi)_Q^{\frac{1}{2}} \\ &\le Ch^{\frac{1}{2}} \|u_h\|_{L^\infty(Q)} \|\partial_x \phi\|_{L^\infty(I;H^1(\Omega))}. \end{aligned}$$

Considering the upper bounds for I, II and III in combination with Lemma 3.1, 3.2 and 3.5 it follows that for each $\varepsilon > 0$ there exists $h_0 > 0$ such that for all $h < h_0$

$$\left| (u_h, \partial_t \phi)_Q + \left(\frac{u_h^2}{2}, \partial_x \phi \right)_Q \right| < \varepsilon$$

and the claim follows by the convergence

$$|(u_h(0) - u_0, \phi)| = |(u_h(0) - u_0, \phi - \mathcal{I}_h \phi)| \le Ch \|u_0\|_{L^1(\Omega)} \|\phi\|_{W^{1,\infty}(\Omega)} \to 0$$

as $h \to 0$ for all $\phi \in C^\infty(\bar{\Omega})$. □

THEOREM 3.8. *(Satisfaction of entropy inequalities) Let u_h be the finite element solution of (2.2) with a semilinear form (3.1) satisfying the strong DMP-property and $g = 0$, then there holds*

$$\lim_{h \to 0} ((\eta(u_h), \partial_t \phi)_Q - (\psi(u_h), \partial_x \phi)_Q) \le 0$$

for the strictly convex entropy $\eta(s) = s^2$ and associated entropy flux ψ such that $\psi'(s) = f'(s)\eta'(s)$ and for all testfunctions $\phi \in C_0^\infty(Q)$, $\phi \ge 0$.

PROOF. If $\eta(s) = s^2$ we note that $\eta'(u_h) = 2u_h$ and hence Lemma 3.6 holds for the product $\eta'(u_h)\phi$. We proceed by rewriting the entropy inequality on weak form as a sum of residuals accounting for the quadrature error in the discrete scalar product and Galerkin orthogonality.

$$\begin{aligned} &|-(\eta(u_h), \partial_t \phi)_Q - (\psi(u_h), \partial_x \phi)_Q + (\phi \epsilon(u_h) \partial_x u_h, \eta''(u_h) \partial_x u_h)_Q| \\ &= \Big| -\int_0^T (\partial_t u_h, \eta'(u_h)\phi)_h \mathrm{d}t - \int_0^T \int_\Omega (\eta(u_h)\partial_t \phi - \mathcal{I}_h(\eta(u_h)\partial_t \phi)) \; \mathrm{d}x \, \mathrm{d}t \\ &\quad + \left(\partial_x \frac{u_h^2}{2}, \eta'(u_h)\phi \right)_Q + (\epsilon(u_h)\partial_x u_h, \partial_x(\phi \eta'(u_h)))_Q - (\epsilon(u_h)\partial_x u_h, \eta'(u_h)\partial_x \phi)_Q \Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \Big| \int_0^T \sum_i (\partial_t u_h(x_i)\eta'(u_h(x_i))\phi(x_i) - \partial_t u_h(x_i)\mathcal{I}_h(\eta'(u_h(x_i))\phi(x_i)))h \,\mathrm{d}t\Big| \\
&\quad + \Big| \int_0^T \int_\Omega (\eta(u_h)\partial_t\phi - \mathcal{I}_h(\eta(u_h)\partial_t\phi)) \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\quad + \left| \left(\partial_x \frac{u_h^2}{2}, \eta'(u_h)\phi - \mathcal{I}_h(\eta'(u_h)\phi) \right)_Q \right| \\
&\quad + |(\epsilon(u_h)\partial_x u_h, \partial_x(\phi\eta'(u_h) - \mathcal{I}_h(\eta'(u_h)\phi)))_Q| + |(\epsilon(u_h)\partial_x u_h, \eta'(u_h)\partial_x\phi)_Q| \\
&= I + II + III + IV + V.
\end{aligned}
$$

Consider I–V term by term. We estimate term I by applying Lemma 3.6, equation (3.6),

$$
\begin{aligned}
I &\leq \Big(\int_0^T \sum_i |\partial_t u_h(x_i)| h \,\mathrm{d}t \Big) \|\eta'(u_h)\phi - \mathcal{I}_h(\eta'(u_h)\phi)\|_{L^\infty(Q)} \\
&\leq C\|\partial_t u_h\|_{L^1(Q)} \|\eta'(u_h)\|_{L^\infty(Q)} h \|\phi\|_{L^\infty(I;W^{1,\infty}(\Omega))}.
\end{aligned}
$$

For term two we use standard interpolation in L^1 to obtain

$$
\begin{aligned}
II &\leq Ch\|\partial_x(\eta(u_h)\partial_t\phi)\|_{L^1(Q)} \\
&\leq h(\|\eta'(u_h)\|_{L^\infty(Q)}\|\partial_x u_h\|_{L^1(Q)}\|\partial_t\phi\|_{L^\infty(Q)} + \|\eta(u_h)\|_{L^\infty(Q)}\|\partial_t\partial_x\phi\|_{L^1(Q)}).
\end{aligned}
$$

For term III once again we apply Lemma 3.6, equation (3.6),

$$
III \leq \|u_h\|_{L^\infty(Q)}\|\partial_x u_h\|_{L^1(Q)}\|\eta'(u_h)\|_{L^\infty(Q)} h \|\phi\|_{L^\infty(I;W^{1,\infty}(\Omega))}
$$

Upper bounds for the remaining two terms are obtained by

$$
\begin{aligned}
IV &\leq \big\|\epsilon(u_h)^{\frac{1}{2}}\partial_x u_h\big\|_Q \big\|\epsilon(u_h)^{\frac{1}{2}}\partial_x((\eta'(u_h)\phi) - \mathcal{I}_h(\eta'(u_h)\phi))\big\|_Q \\
&\leq C\|\eta'(u_h)\|_{L^2(Q)} h^{\frac{1}{2}} \|\phi\|_{L^\infty(I;W^{1,\infty}(\Omega))}
\end{aligned}
$$

and

$$
V \leq C\big\|\epsilon(u_h)^{\frac{1}{2}}\partial_x u_h\big\|_Q \|\eta'(u_h)\|_{L^\infty(Q)} h^{\frac{1}{2}} \|\phi\|_{L^\infty(I;H^1(\Omega))}.
$$

Here we used the stability estimate (3.1) in both IV and V combined with Lemma 3.6 equation (3.5) for IV. It follows by the estimates of Lemmas 3.1, 3.2 and 3.5 that for each $\varepsilon > 0$ there exists $h_0 > 0$ such that for $h < h_0$

$$
|-(\eta(u_h),\partial_t\phi)_Q - (\psi(u_h),\partial_x\phi)_Q + (\phi\epsilon(u_h)\partial_x u_h, \eta''(u_h)\partial_x u_h)_Q| < \varepsilon.
$$

Hence

$$
\lim_{h\to 0} (-(\eta(u_h),\partial_t\phi)_Q - (\psi(u_h),\partial_x\phi)_Q + (\phi\epsilon(u_h)\partial_x u_h, \eta''(u_h)\partial_x u_h)_Q) = 0
$$

and since the third term in the left hand side is positive we may conclude. ◻

REMARK 3.3. Since the flux function is strictly convex, one strictly convex entropy function is enough for uniqueness as shown by Panov [13]. We conclude that u_h converges to the unique entropy solution of Burgers equation as $h \to 0$.

4 A nonlinear artificial viscosity guaranteeing the DMP-property and the von Neumann artificial viscosity.

In this section we will construct an artificial viscosity that satisfies the hypotheses (H1)–(H3) and that has a weak consistency property. This leads to a class of forms $a_h(\cdot\,;\cdot)$ that are locally Lipschitz continuous and have a DMP-property. Then we recall the classical von Neumann–Richtmyer artificial viscosity and show that although it does not enter the above framework it has a weak DMP-property and thus guarantees that the solution of the conservation law remains positive for all time.

THEOREM 4.1. *Let the semilinear form of (2.2) be defined by*

$$a_{h,1}(u_h;v_h) = -\left(\frac{u_h^2}{2}, \partial_x v_h\right)_\Omega + \sum_K (\epsilon_{1,K}(u_h)\partial_x u_h, \partial_x v_h)_K,$$

with

$$\epsilon_{1,K}(u_h) = \nu h \|u_h\|_{\infty,K} \left(\max_{x\in\partial K} \frac{|[\![\partial_x u_h(x)]\!]|}{2\{|\partial_x u_h(x)|\}}\right)^p.$$

If $\{|\partial_x u_h(x)|\} = 0$ for one node $x \in \partial K$ we define $\epsilon_{1,K}(u_h) = \nu h \|u_h\|_{\infty,K}$. Under these assumptions $a_{h,1}(\cdot\,;\cdot)$ is locally Lipschitz continuous and enjoys the strong DMP-property for $\nu > \frac{1}{2}$ and for all $p \geq 0$.

PROOF. In this case Lipschitz continuity follows by the Lipschitz continuity of $\|u_h\|_{\infty,K}$ and the Lipschitz continuity of the function

$$f(x,y) = \frac{|x-y|}{||x|+|y||}, \quad \text{when } x \neq 0 \text{ or } y \neq 0.$$

We assume that there is a local minimum in the node x_i clearly then

$$\max_{x\in\partial K} \frac{|[\![\partial_x u_h(x)]\!]|}{2\{|\partial_x u_h(x)|\}} = \frac{|[\![\partial_x u_h(x_i,t)]\!]|}{2\{|\partial_x u_h(x_i,t)|\}} = 1$$

for both elements K, K' in the support of v_i. By the fact that $h\partial_x u_h \partial_x v_h = -|\partial_x u_h|$ in Ω_i we may deduce that

$$\begin{aligned} a_{h,1}(u_h;v_i) &= (u_h\partial_x u_h, v_i) - \nu \sum_{K\subset\Omega_i} h\|u_h\|_{\infty,K}|\partial_x u_h|_K \\ &\leq \left(\frac{1}{2}-\nu\right) \sum_{K\subset\Omega_i} h\|u_h\|_{\infty,K}|\partial_x u_h|_K. \end{aligned} \tag{4.1}$$

The proof for a local maximum is similar. □

The viscosity ν has to be chosen large enough so as to make the viscous term dominate the flux term at local extrema. On the other hand, p appears as a completely free parameter. Since $f(x,y) \leq 1$ for all x, y with equality at local extrema we see that a large value of p will result in small artificial viscosity away from local extrema. The size of p determines the order of the method: $p = 0$ corresponds to the classical linear artificial viscosity method giving first order accuracy, whereas $p \geq 1$ will give a high order scheme away from local extrema. The higher the value of p the less artificial viscosity is introduced. In the limit $p \to \infty$ the artificial viscosity $\epsilon_{1,K}$ will be 1 in elements where one node is a local extremum and 0 in all other elements. This limit behavior is in some sense optimal, corresponding to the minimal stabilization necessary for the theory of the previous section to apply. However it is known that strongly localized shock capturing terms lead to very ill-conditioned nonlinear systems. This is reflected in the analysis by the fact that the constant in the Lipschitz continuity explodes as $p \to \infty$.

REMARK 4.1. Note that the term $\epsilon_{1,K}$ is strongly related to flux-limiter schemes adding first order artificial viscosity on local extrema only. In fact in the limit of infinite p and on linear model problems with constant transport coefficient one may show that on uniform grids the proposed finite element method corresponds to an upwind scheme at local extrema and a centered difference scheme away from extrema.

We will now consider two different forms of artificial viscosities that are not strong enough to make the semi-linear form have the strong DMP-property, however they guarantee a certain weak DMP-property leading to a proof of positivity of the solutions in the approximating sequence $\{u_h\}_h$. The first form considered is the von Neumann–Richtmyer artificial viscosity [17] and the second is a higher order generalization obtained by replacing the gradient by the jump of the gradient in the viscosity coefficient.

LEMMA 4.2. *Let the semilinear form of (2.2) be defined by*

$$a_{h,i}(u_h; v_h) = -\left(\frac{u_h^2}{2}, \partial_x v_h\right)_\Omega + \sum_K (\epsilon_{i,K}(u_h)\partial_x u_h, \partial_x v_h)_K,$$

with either $i = 2$ (von Neumann/Richtmyer)

$$\epsilon_{2,K}(u_h) = \nu h^2 |\partial_x u_h|, \qquad \textit{or, } i = 3 \qquad \epsilon_{3,K}(u_h) = \nu h_K^2 \max_{x_i \in \partial K} |[\![\partial_x u_h]\!]|_{x_i}|.$$

Then $a_{h,i}(\cdot;\cdot)$, $i = 1, 2$ are locally Lipschitz continuous and satisfy the weak DMP-property for $\nu > \frac{1}{2}$, for u_h that are negative on subdomains containing no more than two nodes.

PROOF. The proof for $\epsilon_{2,K}(u_h)$ is similar to the proof of Theorem 4.1 using the fact that $|u_h| \leq h|\partial_x u_h|$ at one of the elements in which the local minimum is taken, since $u_h(x_i) < 0$. Note that at a local extremum $\epsilon_{3,K}(u_h) \geq \epsilon_{2,K}(u_h)$ whereas away from local extrema $\epsilon_{3,K}(u_h) < \epsilon_{2,K}(u_h)$ by which the result carries over to $\epsilon_{3,K}(u_h)$. $\square$

REMARK 4.2. The Lemma 4.2 is not strong enough to guarantee convergence of the approximating sequence to entropy solutions using the previous analysis,

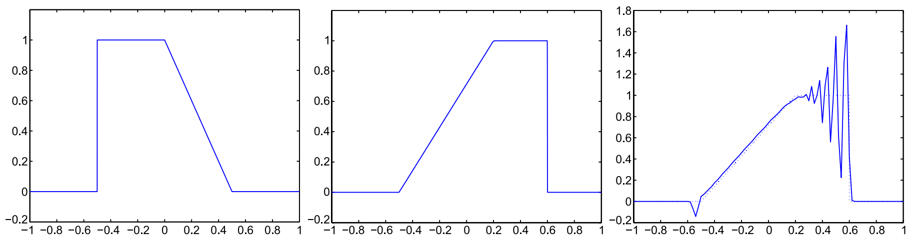

Figure 5.1: From left to right: the initial data u_0, the exact solution at $T = 0.7$ and the standard Galerkin solution at $T = 0.7$.

but it will ensure that solutions with positive initial data will remain positive for all times. Note that $\epsilon_2(u_h)$ is a close relative to the Smagorinsky model for turbulent viscosity (see [14]) and $\epsilon_3(u_h)$ is its higher order generalization. A possible interpretation of the positivity result is that the Smagorinsky model guarantees that the numerical scheme is diffusion dominated close to stagnation points.

Corollary 4.3. *Let u_h be the discrete solution of (2.2) with the semilinear form $a_i(\cdot\,;\cdot)$, $i = 1, 2$. If $u_h(\cdot, 0) \geq 0$ then $u_h(\cdot, t) \geq 0$ for $t > 0$.*

Proof. Clearly u_h can go negative simultaneously in at most two consecutive nodes. Assume therefore that $u_h(x_i) = -\varepsilon$ (with $\varepsilon > 0$) is a local minimum with one neighbour positive. By the weak DMP-property $\partial_t u_h(x_i) > 0$ and since ε is arbitrarily small we may conclude. □

5 Numerical results.

We consider the Burgers' equation on the domain $\Omega = (-1, 1)$ with periodic boundary conditions. The initial data and the exact solution at final time $T = 0.7$ are presented in Figure 5.1 (left and middle graphic). The equations are discretized using 100 elements in space and integrated using the explicit Euler scheme and a very small timestep (CFL $= 0.01$, $k =$ CFL $\times\, h$) so that the error due to time discretization is assumed negligeable. In Figure 5.1, right plot, we show the strongly oscillating solution of the standard Galerkin method. The solutions using the different methods discussed above are presented in Figure 5.2. In the leftmost graphic of Figure 5.2 we give the solution using the standard upwind method for comparison. As predicted by theory, the von Neumann–Richtmyer artificial viscosity ($\epsilon_{2,K}$) yields positive solutions (middle, Figure 5.2), but with spurious oscillations on the shock front. The results when using $\epsilon_{2,K}$ and $\epsilon_{3,K}$ are very similar on this test case and we only give the former ones. The shock-capturing stabilization $\epsilon_{1,K}$ (right, Figure 5.2) on the other hand yields oscillation free solutions and the shock is resolved with one interior point.

In Figure 5.3 (left graphic) the convergence behavior in h is shown on four consecutive meshes, $N = 100$, $N = 200$, $N = 400$ and $N = 800$. Then, in the right graphic of Figure 5.3, the behavior of the approximate solution with

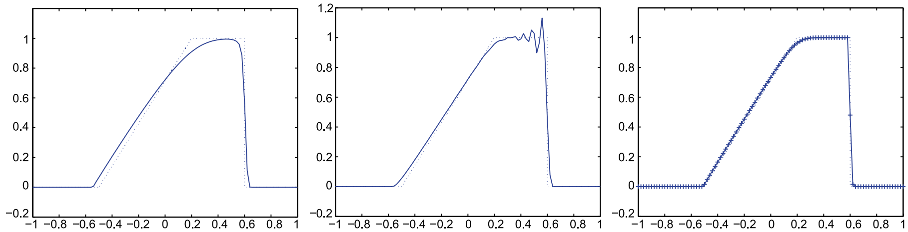

Figure 5.2: From left to right: the standard upwind method, the von Neumann –Richtmyer artificial viscosity ($\nu = 0.5$) of Lemma 4.2, and the DMP-satisfying form of Lemma 2.1 with $\nu = 0.5$, $p = 100$.

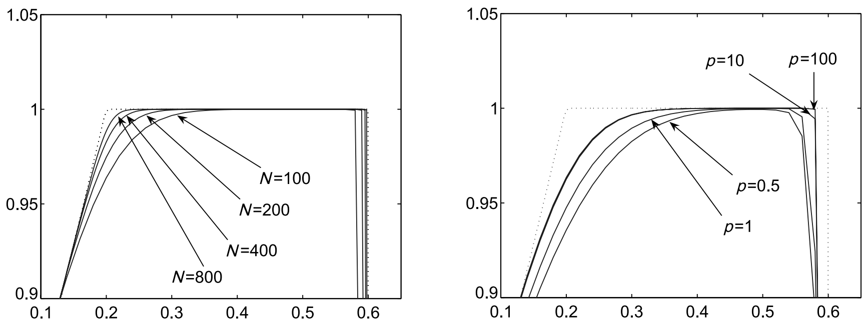

Figure 5.3: Left: Convergence of the approximations with $\nu = 0.5$, $p = 100$ using $N = 100$, $N = 200$, $N = 400$ and $N = 800$; Right: Dependence of the approximate solution on p ($\nu = 0.5$, $N = 100$), $p = 0.5$, $p = 1$, $p = 10$ and $p = 100$.

varying p is investigated. A zoom of the upper part of the solution is shown for a sequence of approximations obtained using different values on p. The most diffused solution corresponds to $p = 0.5$ and then with increasing resolution: $p = 1$, $p = 10$ and $p = 100$. The solutions using $p = 10$ and $p = 100$ are distinguishable only at the shock tip. Hence choosing $p > 10$ does not have a strong influence on solution quality in this example. Finally in Figure 5.4 we study the robustness of the method with respect to different values of ν and CFL using $\epsilon_{1,K}(u_h)$ (the time discretization has not been investigated theoretically, but we give this numerical comparison for completeness). In the left graphic of Figure 5.4 a zoom of the upper part of the shock is presented for an approximate solution obtained with $\nu = 0.49$ and various values of p. The solution clearly violates the DMP for $p > 1$ showing that the limit value of ν is sharp (note however the scale on the y-axis). As can be expected lower values on p are more robust with respect to variations in ν. Finally in the right graphic of Figure 5.4 we consider the effect of varying CFL condition for $p = 1$. The cases CFL $= 0.9$, CFL $= 0.5$ and CFL $= 0.1$ are considered. As expected CFL $= 1$ was the limit value for stability. Similar resolution of the shock is observed for CFL $= 0.5$ and CFL $= 0.1$ the rarefaction wave on the other hand is better resolved for

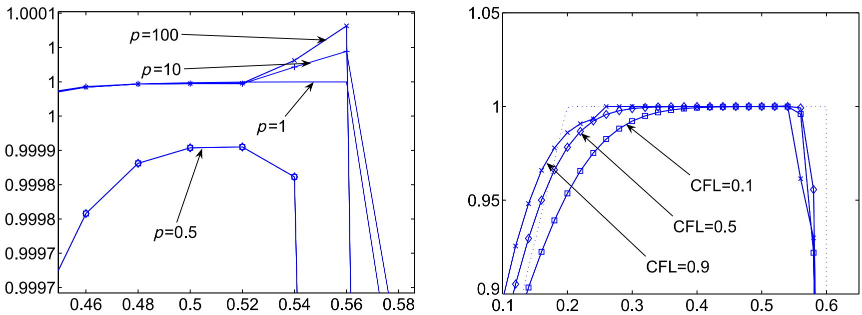

Figure 5.4: Left: Zoom of function crest showing the violation of the DMP using $\nu = 0.49$ and $p = 0.5$, $p = 1$, $p = 10$ and $p = 100$; Right: Zoom of function crest for different values of the CFL $= \frac{k}{h}$ ($N = 100$, $p = 1$).

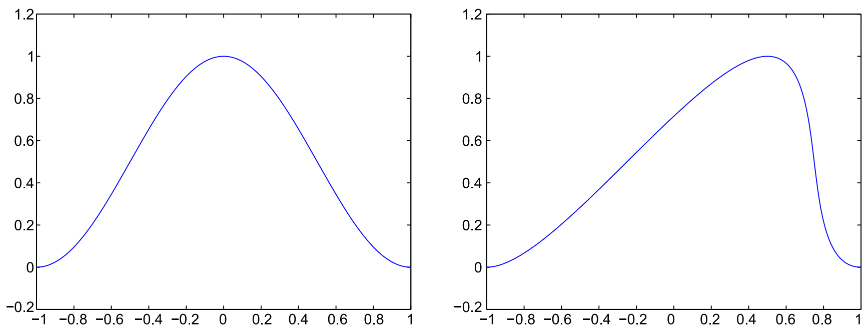

Figure 5.5: Left: the smooth initial data u_0, right: the exact solution at $T = 0.5$.

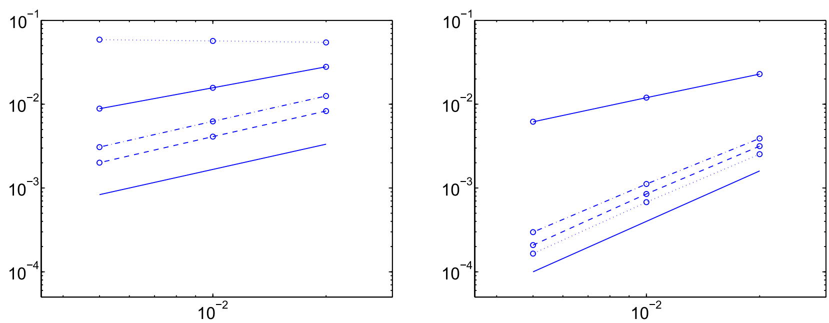

Figure 5.6: Log-log plots of the h-convergence of the error in norm L^1. Left plot: the non smooth solution given by the initial data of Figure 5.1. Right plot: the smooth solution given by the initial data (5.1). The methods are as follows: $p = 0$ (full line with circles), $p = 1$ (dash-dot line with circles), $p = 10$ (dashed line with circles) and standard Galerkin, i.e. $\epsilon(u_h) = 0$, (dotted line with circles). In the left plot we also give the the slope of $O(h)$ (full line, without markers) and in the right plot the slope of $O(h^2)$ (full line, without markers).

CFL $= 0.5$. It was noted during the numerical experiments that higher p required lower CFL for the scheme to satisfy the DMP. In Figure 5.6 (left plot) we give the convergence of the error in the L^1-norm. As expected the error is $O(h)$ for all values of p due to the discontinuous solution. Observe that the standard Galerkin method does not converge.

5.1 A numerical example with smooth solution.

In this case our aim is to investigate the global order of convergence of our method for different values of p on a problem with smooth solution. To this end we consider the initial data

$$u_0(x) = \frac{1}{2}\left(\cos(\pi x) + 1\right). \tag{5.1}$$

We compute the solution on three consecutive meshes with $N = 50$, $N = 100$ and $N = 200$ respectively. The final time, $T = 0.5$, was chosen just before the standard finite element method exhibits the first signs of instability on the coarsest mesh. A reference solution was computed using the unstabilized finite element method on a mesh with $N = 1000$. In all computations we chose CFL $=$ 0.001 to avoid any influence of the time discretization. In Figure 5.5 we report the initial data and the solution at time $T = 0.5$. In Figure 5.6 (right plot) we report log-log plots for the convergence of the error in the norm L^1 for the cases $p = 0$ (full line with circles), $p = 1$ (dash-dot line with circles) and $p = 10$ (dashed line with circles). As a reference we also give the convergence of the unstabilized finite element method resulting in a centered finite difference scheme (dotted line with circles) and the slope of $O(h^2)$ (full line, without markers). For $p = 0$ we report first order convergence in space, which is expected since this corresponds to standard artificial viscosity. For $p = 1$ and $p = 10$ the convergence is $O(h^2)$ with a slightly smaller error constant for the larger value on p. The standard finite element method has the smallest constant and also exhibits second order convergence. The difference in the constant between the case $p = 1$ and the standard finite element scheme is less than a factor 2.

6 Conclusion.

In a simple framework we have pointed out similarities between some artificial viscosity methods and recent finite element stabilization techniques for convection-dominated flow problems. The nonlinear artificial viscosity is of order h close to local maxima and minima, thereby assuring a discrete maximum principle. This local perturbation does not seem to affect the global convergence order for smooth solutions. The present analysis holds only for the space of piecewise affine continuous functions and in one space dimension. In the case of higher order polynomial spaces the question of discrete maximum principle is completely open (also for elliptic problems) and the present type of simple artificial viscosity using only fluctuations of the gradient over element edges can hardly be expected to work, since spurious frequencies may be present both in the polynomial spectrum and as derivative jumps on element boundaries. For a recent analysis of a similar (linear) stabilized method applied to a linear model problem in the high order case we refer to [5]. The main obstacle for the extension of the present work to several space dimensions is the construction of artificial viscosity terms that are both Lipschitz continuous and makes the semilinear form enjoy the strong DMP-property.

We hope that these arguments will provide some further insight in the close relationsship between high resolution methods and stabilized finite element methods in flow computations.

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